

Boolean graphs are unmixed and vertex decomposable*

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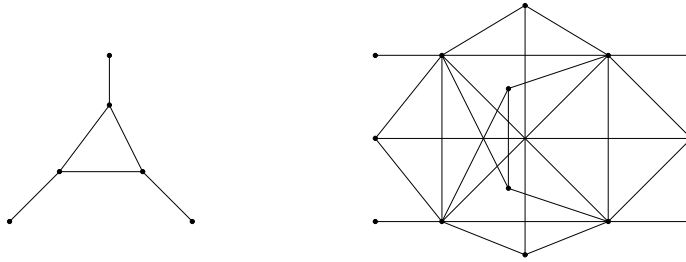
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Abstract. For each Boolean graph B_n , it is proved that both B_n and its complement graph $\overline{B_n}$ are vertex decomposable. It is also proved that B_n is an unmixed graph, thus it is also Cohen-Macaulay.

Key Words: Boolean graph, vertex decomposable, unmixed, generalized blow up graph

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Throughout, let $[n] = \{1, 2, \dots, n\}$ and let $2^{[n]}$ be the power set of $[n]$. Recall from [9] that a finite Boolean graph, denoted by B_n , is a graph defined on the vertex set $2^{[n]} \setminus \{[n], \emptyset\}$, with M adjacent to N if $M \cap N = \emptyset$; see also [7]. In the following, we list the graphs B_3 and B_4 in diagrams (note that the center crossed in B_4 is not a vertex):



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Throughout, we use 421 to denote the vertex $\{1, 2, 4\}$ of $V(B_4)$. We assume

$$n > n - 1 > \cdots > 2 > 1,$$

and use the pure lexicographic order on the vertices of $V(B_n)$, eg., $5421 > 5321$ in $V(B_6)$.

The original purpose of this work is a try to study the combinatorial property of the finite Boolean graph B_n , such as shellability or Cohen-Macaulayness. In the process, we find that both B_n and its complement graph $\overline{B_n}$ have nice properties, and so is the related (pure) skeleton complexes and the related Alexander dual complex.

This paper is organized as follows. In section 1, we recall some basic concepts, facts and backgrounds from combinatorial commutative algebra. In section two, we first prove that B_n is an unmixed graph, and then give a complicated algorithm to check that B_n is also vertex decomposable. In section 3, we study the properties of the complement graph $\overline{B_n}$. In section 4, we have a preliminary study on the unmixed property of a blow up of a Boolean graph.

1 Preliminaries

In this part, we recall some definitions and results in combinatorial commutative algebra. For more details without mention, one can refer to the recent monographs, e.g., [16, 6].

Recall that a simplicial complex Δ is a subset of the power set $2^{[n]}$ of $[n]$, such that Δ is hereditary and, all singletons x ($1 \leq x \leq n$) are in Δ . x is called a vertex of the complex Δ . Recall that

$$\Delta \setminus x = \{F \in \Delta \mid x \notin F\}, \quad lk_{\Delta}(x) = \{F \in \Delta \mid x \notin F, F \cup \{x\} \in \Delta\}.$$

Definition 1.1. *Let Δ be a simplicial complex over $[n]$. If one of the following inductive condition is satisfied, then Δ is called **vertex decomposable**:*

- (1) Δ is a simplex, or
- (2) There is a vertex x such that the following requirements are fulfilled
 - (α) Both $\Delta \setminus x$ and $lk_{\Delta}(x)$ are vertex decomposable.
 - (β) No facet of $lk_{\Delta}(x)$ is a facet of $\Delta \setminus x$, or equivalently,

$$\Delta \setminus x = \langle \{F \mid x \notin F \in \mathcal{F}(\Delta)\} \rangle.$$

Such a vertex x satisfying conditions (α) and (β) is called a **shedding vertex** of Δ . If x only satisfies the second condition, then we call it a *weak shedding vertex*.

Recall the following implications for nonpure simplicial complexes:

$$shifted \implies vertex\ decomposable \implies shellable \iff strongly\ shellable$$

Recall the following implications for simplicial complexes:

$$matroid \implies vertex\ decomposable, \text{ and } pure \implies pure\ shellable$$

$$\implies \text{constructible} \implies \text{Cohen} - \text{Macaulay} \implies \text{pure}.$$

For the definition of strongly shellable, see [4]. By [4], if Δ is strongly shellable, then both I_{Δ^\vee} and $I(\Delta)$ have linear quotients, where $\Delta^\vee = \{[n] \setminus F \mid F \notin \Delta\}$ and is called the *Alexander dual complex* of Δ . Note that in [4], counterexamples are given to show that *there is no implication between the concepts vertex decomposable and strongly shellable*.

The following result is well-known. Note that a similar result holds true for each of the following properties: shifted, strongly shellable, shellable, Cohen-Macaulay, sequentially Cohen-Macaulay.

Proposition 1.2. *Let Δ_1 and Δ_2 be complexes over $[n] = [1, n]$ and $[n+1, n+m]$ respectively. Then the join complex $\Delta_1 * \Delta_2$ is vertex decomposable if and only if both complexes Δ_1 and Δ_2 are vertex decomposable.*

For a graph G , recall that the edge ideal $I(G)$ is identical with the Stanley-Reisner ideal I_{Δ_G} of the clique complex Δ_G of the complement graph \overline{G} . Recall that a graph G is called *vertex decomposable* (Cohen-Macaulay, or shellable, or unmixed, respectively) if the simplicial complex Δ_G has the corresponding property. Thus we have

Corollary 1.3. ([17, Lemma 20]) *A graph G is vertex decomposable if and only if all connected components of G are vertex decomposable.*

For a vertex x in a graph G , let $N_G[x] = N_G(x) \cup \{x\}$, the closed neighborhood of x in G . The following is a translation of vertex decomposable of a simplicial complex in the language of a graph:

Definition 1.4. ([17, Lemma 4]) *A graph G is called vertex decomposable if either it has no edges, or else has some vertex x such that we have as follows:*

- (1) *Both $G \setminus N_G[x]$ and $G \setminus x$ are vertex decomposable.*
- (2) *For every independent set S in $G \setminus N_G[x]$, there exists some $y \in N_G(x)$ such that $S \cup \{y\}$ is independent in $G \setminus x$.*

The following result tells how to construct new and large vertex decomposable graphs from known ones:

Proposition 1.5. ([12, Proposition 2.3]) *Let G_1, \dots, G_n be finite graphs, and assume $|V(G_i)| \geq 2$, $V(G_i) \cap V(G_j) = \emptyset$ for all $i \neq j$. For a graph G with n vertices x_i , let $G(G_i \mid 1 \leq i \leq n)$ be a graph obtained by attaching x_i with a vertex in G_i . (x_i is called a *gluing vertex*.)*

If the graphs G_1, \dots, G_n are vertex decomposable and each gluing vertex x_i is a shedding vertex of G_i , then $G(G_i \mid 1 \leq i \leq n)$ is vertex decomposable.

Chordal graphs are an important class of vertex decomposable graphs. Recall that a graph is called *chordal*, if all cycles of four or more vertices have a chord, which is an edge

that is not part of the cycle but connects two vertices of the cycle. Adam Van Tuyl, Rafael H. Villarreal in [15] proved that all chordal graphs are (nonpure) shellable. Woodroffe in [17] proved further that a chordal graph is vertex decomposable and later, generalized the idea to clutters in [18].

Recall the following theorem, which contains important results in the algebraic combinatorics of a chordal graph:

Theorem 1.6. *Let G be a graph and \overline{G} the complement graph of G . Then the following three conditions are equivalent:*

- (1) \overline{G} is chordal.
- (2) (Fröberg [3]) The edge ideal $I(G)$ of G has a linear resolution.
- (3) (Lyubeznik [8]) The cover ideal $I_c(G)$ is Cohen-Macaulay, where $I_c(G)$ is the edge ideal of the clutter consisting of all minimal vertex covers of G .

Recall that a *simplicial vertex* of a graph is a vertex v such that the neighbourhood $N(v)$ is a clique. Recall the following main theorem (by Dirac) characterizing chordal graphs:

Theorem 1.7. *A graph G is chordal if and only if every induced subgraph of G has a simplicial vertex.*

2 Boolean graphs

Recall that a *vertex cover* C of a graph G is a subset of the vertex set $V(G)$ such that

$$C \cap \{i, j\} \neq \emptyset, \forall \{i, j\} \in E(G).$$

A vertex cover is also called a *dominating set* of G , while the *dominating number* of G is the least of cardinalities of all minimal vertex covers. Recall that a graph G is said to be *unmixed*, if all minimal vertex covers of G have a same cardinality. It is known that G is unmixed if and only if the clique simplicial complex of \overline{G} is unmixed, while a Cohen-Macaulay graph is always unmixed. Recall also that C is a minimal vertex cover if and only if $V(G) \setminus C$ is a maximal independent vertex set of G .

Now we give the first main result of this paper:

Theorem 2.1. *Let $n \geq 1$, and let G be the Boolean graph B_n . Then G is unmixed.*

Proof: We give a proof by considering independent vertex sets. Note that a vertex subset V_0 is a minimal vertex cover of G if and only if V_0^C is a maximal independent vertex set of G , where $V_0^C = V(G) \setminus V_0$. In the following, we proceed to prove that all maximal independent set of G have the same cardinality of $2^{n-1} - 1$. In fact, a subset $\mathfrak{d} = \{b_1, b_2, \dots, b_t\}$ of $2^{[n]}$ is an independent vertex set of G , iff $b_i \cap b_j \neq \emptyset$ holds for distinct

b_i, b_j in \mathfrak{d} . As the cardinality of the vertex set $V(G)$ is $2^n - 2$, we can distribute $V(G)$ into two parts $\{b_1, b_2, \dots, b_{2^{n-1}-1}\}$ and $\{b_1^c, b_2^c, \dots, b_{2^{n-1}-1}^c\}$. For any independent vertex set \mathfrak{d} of G with $|\mathfrak{d}| < 2^{n-1} - 1$, we claim that more vertices can be added to \mathfrak{d} to obtain a larger independent vertex set, until the cardinality reaches $2^{n-1} - 1$. For this, observe first that for any vertex b in $V(G)$, the complement b^c is also in $V(G)$, and this is a one to one corresponding. Thus, the cardinality of \mathfrak{d} is no larger than $2^{n-1} - 1$. Second, for any independent vertex set $\mathfrak{d} = \{b_1, \dots, b_t\}$ of G with $|\mathfrak{d}| = t < 2^{n-1} - 1$, clearly there exists b_{t+1} such that $\{b_{t+1}, b_{t+1}^c\} \cap \mathfrak{d} = \emptyset$. If neither $\mathfrak{d} \cup \{b_{t+1}\}$ nor $\mathfrak{d} \cup \{b_{t+1}^c\}$ is independent in the graph G , then there are $b_i \in \mathfrak{d}$ and $b_j \in \mathfrak{d}$, such that $b_i \cap b_{t+1} = \emptyset$ and $b_j \cap b_{t+1}^c = \emptyset$. Then $b_i \subseteq b_{t+1}^c$ and $b_j \subseteq b_{t+1}$, contradicting $b_i \cap b_j \neq \emptyset$. This shows that the graph G is unmixed. ■

Note that B_n is not chordal when $n \geq 4$, since $1 - 23 - 14 - 2 - 1$ is a cycle and it has no chord. Note also that a Boolean graph B_n is not matroidal for any $n \geq 3$. In fact, the clique complex of the complement graph $\overline{B_n}$ is far from being a matroid in general, as the following example shows:

Example 2.2. *The clique complex of the complement graph $\overline{B_3}$ is*

$$\Delta = \langle \{1, 12, 13\}, \{2, 12, 23\}, \{3, 13, 23\}, \{12, 13, 23\} \rangle.$$

Note that the vertex set of Δ is $2^{[3]} \setminus \{[3], \emptyset\}$, so if we take a subset of it as $W = \{1, 2, 12, 13\}$, then $\Delta_W = \langle \{1, 12, 13\}, \{2, 12\} \rangle$. Since the induced subcomplex Δ_W is not pure, by [14, Proposition 3.1], the complex Δ is not a matroid.

Next we want to prove that all Boolean graphs are vertex decomposable. In order to do so, recall that a vertex u in a graph G is said to have a *whisker*, if there is an end vertex adjacent to u ([16, Definition 7.3.10]). We observe the following:

Lemma 2.3. *Any vertex in a graph G with whiskers is a weak shedding vertex.*

Proof: Let d be an end vertex adjacent to u . Clearly, $u \notin G \setminus N_G[u]$, $d \in N_G(u)$, and any independent set of $G \setminus N_G[u]$ can be extended to a larger independent vertex set $D \cup \{d\}$ in $G \setminus u$. Thus u is a weak shedding vertex of G . ■

Note that each of the vertices $1, \dots, n$ has a whisker in B_n . If let

$$G_1 = B_n \setminus 1 \setminus 2 \setminus \dots \setminus n,$$

then each ji has a whisker in the graph G_1 for all $1 \leq i < j \leq n$; If let

$$G_2 = G_1 \setminus 12 \setminus 13 \setminus \dots \setminus n - 1n,$$

then every kji ($n \geq k > j > i \geq 1$) has a whisker in the graph $G_2; \dots$. Thus in order to show that B_n is vertex decomposable, we will choose

$$1, \dots, n; 21, \dots, nn - 1; 321, \dots$$

as a sequence of weak shedding vertices. Note also that $G \setminus N_G[v] \subseteq G \setminus v$.

In the following, we present a weak shedding vertex order to prove the second main result of this paper:

Theorem 2.4. *For any $n \geq 1$, let $G = B_n$ be the Boolean graph. Then G is vertex decomposable, hence Cohen-Macaulay.*

Proof: Let

$$G_{n+1} = B_n, G_n = G_{n+1} \setminus n, G_{n-1} = G_n \setminus n-1, \dots, G_1 = G_2 \setminus 1. \quad (1)$$

Note that $G \setminus N_G[n] = \{A \cup \{n\} \mid A \in V(B_{n-1})\}$, and that it is a discrete graph, hence vertex decomposable by Corollary 1.3. Note also that

$$\begin{aligned} G_n \setminus N_{G_n}[n-1] &= G \setminus N_G[n-1] \setminus n, \\ G_{n-1} \setminus N_{G_{n-1}}[n-2] &= G \setminus N_G[n-2] \setminus n \setminus n-1, \\ &\dots\dots\dots \\ G_2 \setminus N_{G_2}[1] &= G \setminus N_G[1] \setminus n \setminus n-1 \setminus \dots \setminus 2, \end{aligned}$$

thus they are all discrete graphs and hence, vertex decomposable. Note that each of $\{i\}$ is a weak shedding vertex of the graph G_{i+1} , thus by Definition 1.4, the graph G is vertex decomposable if and only if the subgraph G_1 is vertex decomposable.

In order to see that the graph G_1 is vertex decomposable, let

$$\begin{aligned} G_{nn-1} &= G_1 \setminus nn-1, G_{nn-2} = G_{nn-1} \setminus nn-2, \dots, G_{n1} = G_{n2} \setminus n1 \\ G_{n-1n-2} &= G_{n1} \setminus \{n-1, n-2\}, \dots, G_{n-11} = G_{n-12} \setminus \{n-1, 1\} \\ &\dots\dots\dots, \\ G_{32} &= G_{41} \setminus 32, G_{31} = G_{32} \setminus 31, G_{21} = G_{31} \setminus 21 \end{aligned} \quad (2)$$

Note that each vertex ij is a weak shedding vertex of the graph in front of it. Now consider the corresponding $H \setminus N_H[ij]$. Let $H = G_1 \setminus N_{G_1}[nn-1]$. We have $H = (H_1 \setminus nn-1) \cup (\cup_{i \geq 2} H_{2i})$, where

$$H_1 = \{A \in V(B_n) \mid |A| \geq 2, n \in A\}, H_{2i} = \{A \in V(B_n) \mid |A| = i, n-1 \in A, n \notin A\}.$$

Note that both H_1 and $\cup_{i \geq 2} H_{2i}$ are discrete graphs, and that each vertex of H_{22} has a whisker in the graph H (surely, in H_1), thus any linear order of vertices of H_{22} is a weak shedding order of H . Then we delete H_{22} , and consider $H \setminus H_{22}$. Surely, each vertex of H_{23} has a whisker in $H \setminus H_{22}$ (again, in H_1), thus we delete H_{23} from $H \setminus H_{22}$, and continue the discussion, until we reach a forest. This shows that H is vertex decomposable. In a

similar way, we see that each of $G_{n-1i} \setminus N_{G_{n-1i}}[n-1i-1]$ is vertex decomposable. Thus the graph G_1 is vertex decomposable if and only if the graph G_{21} is vertex decomposable.

Now assume $n \geq 6$. In order to see that G_{21} is vertex decomposable, the next step is to consider the sequential deletions:

$$G_{nn-1n-2} =: G_{21} \setminus \{n, n-1, n-2\}, \dots, G_{321} =: G_{421} \setminus \{3, 2, 1\} \quad (3)$$

and the related $H \setminus N_H[ijk]$. In the process, we always take advantage of the vertices with whiskers. For the graph $L = G_{21}$, let $H = L \setminus N_L[nn-1n-2]$. Then

$$V(H) = H_1 \cup (\cup_{i \geq 3} (H_{2i} \cup H_{3i})),$$

where

$$H_1 = \{A \in V(B_n) \mid |A| \geq 3, n \in A, A \neq nn-1n-2\}$$

$$H_{2i} = \{A \in V(B_n) \mid |A| = i, n-1 \in A, n \notin A\}$$

$$H_{3i} = \{A \in V(B_n) \mid |A| = i, n-2 \in A, A \cap \{n, n-1\} = \emptyset.\}$$

Note that the subgraphs induced on each H_i is discrete, and that each vertex of $H_{33} \cup H_{23}$ has a whisker in H , with an adjacent end vertex in H_1 . Thus in order to see that H is vertex decomposable, we delete H_{33} and H_{23} from H , then going on to consider the vertices with whiskers. In this way, we show that the graph G_{21} is vertex decomposable iff G_{321} is vertex decomposable.

We continue this process for both related $H \setminus u$ and $H \setminus N_H[u]$, until it reaches a discrete graph or a forest. In this way, due to the fact that the related $H \setminus N_H[u]$ always has enough weak shedding vertices (actually, vertices which have whiskers), in the end we are able to prove that B_n is actually vertex decomposable.

Finally, it is known that vertex decomposable implies shellability, while pure shellability implies Cohen-Macaulayness. Thus by Theorem 2.1, the graph B_n is Cohen-Macaulay. ■

We remark that very detailed check has been taken when $n = 4, 5, 6$, showing that both B_n and $\overline{B_n}$ are vertex decomposable. In the next section, we will show that the graph $\overline{B_n}$ is also vertex decomposable.

3 The complement graph $\overline{B_n}$

Note that the graph $\overline{B_n}$ is not chordal when $n \geq 4$, since the cycle $21-32-43-41-21$ has no chord. Note also that the graph $\overline{B_n}$ is not matroidal for any $n \geq 3$. In fact, the clique complex Δ of B_3 is not pure.

Nevertheless, the complement graph also has some nice properties, see the following third main result of this paper:

Theorem 3.1. *For any $n \geq 1$, the complement graph \overline{G} of the Boolean graph $G = B_n$ is vertex decomposable.*

Proof: For $n = 3$, the result is clear. In the following, assume $n \geq 4$. Like in the Boolean case, we choose a sequence of weak shedding vertices according to their vertex degree, and we choose it first if it has greater vertex degree. For the vertices of a same degree, we use pure lexicographic order with $n > n - 1 > \dots > 1$. Let

$$G_0 = \overline{B_n}, G_i = G_0 \setminus \{\overline{1}, \dots, \overline{i}\} \quad i = 1, 2, \dots, n$$

where $\overline{1} = 23 \dots n$. Note that

$$G_i \setminus N_{G_i}[\overline{i+1}] = \{i\}, \quad \forall i = 0, 1, \dots, n-1,$$

and clearly condition (2) of Definition 1.4 is fulfilled, hence the graph $\overline{B_n}$ is vertex decomposable if and only if the graph G_n is vertex decomposable. Let

$$\begin{aligned} G_{01} &= G_n, G_{12} = G_{01} \setminus \overline{12}, \dots, G_{1n} = G_{1n-1} \setminus \overline{1n}, \\ G_{23} &= G_{1n} \setminus \overline{23}, G_{24} = G_{23} \setminus \overline{24}, \dots, G_{2n} = G_{2n-1} \setminus \overline{2n}, \\ &\dots\dots\dots \\ G_{n-1n} &= G_{n-2n} \setminus \overline{n-1n}. \end{aligned}$$

Now consider the corresponding sequence $H \setminus N_H[u]$. Note that

$$G_{01} \setminus N_{G_{01}}[\overline{12}] = \{1, 2, 12\} = \overline{B_2} \cup 12,$$

in which 12 is adjacent to all vertices of $\overline{B_2}$, thus $G_{01} \setminus N_{G_{01}}[\overline{12}]$ is vertex decomposable. Since all the corresponding $H \setminus N_H[u]$ have a same structure, they are all vertex decomposable. Note that $3 \in N_{G_{01}}(12)$, and 3 is independent to all vertices of $G_{01} \setminus N_{G_{01}}[\overline{12}]$, thus $\overline{12}$ is a shedding vertex of the graph G_{01} . Similarly, it is easy to see that the sequence

$$\overline{12}, \dots, \overline{1n}, \overline{23}, \dots, \overline{2n}, \dots, \overline{n-1n}$$

is a shedding vertex order. Hence the graph G_n is vertex decomposable if and only if the graph G_{n-1n} is vertex decomposable.

We continue the discussion by letting

$$G_{123} = G_{n-1n} \setminus \overline{123}, G_{124} = G_{123} \setminus \overline{124} \dots, G_{n-2n-1n} = G_{n-3n-1n} \setminus \overline{n-2n-1n}.$$

We also have

$$G_{n-1n} \setminus N_{G_{n-1n}}[\overline{123}] = 2^{[3]} \setminus \dots = \overline{B_3} \cup 123,$$

in which the vertex 123 is adjacent to every vertex of the vertex decomposable graph $\overline{B_3}$. Note also that 4 is shedding vertex. This shows that the graph G_{n-1n} is vertex

decomposable if and only if the graph $G_{n-2n-1n}$ is vertex decomposable. This also verifies that $\overline{B_4}$ is vertex decomposable.

If we continue this process beginning from G_{1234} and ending at $G_{n-3n-2n-1n}$, then we proved the result for $n = 5$.

This completes the verification. Clearly, this proof is a not bad algorithm, just like the proof to Theorem 2.4. ■

Recall that a *skeleton* complex $\Delta^{(0,s)}$ is a subcomplex of Δ , which consists of all faces F of Δ with $|F| \leq s + 1$. Recall that a *pure skeleton* complex $\Delta^{(s,s)}$ is a subcomplex of Δ , which is generated by all faces of Δ of dimension s . Recall that all skeletons and pure skeletons of a shellable complex are shellable.

By the proofs of Theorems 2.4 and 3.1, we have the following

Corollary 3.2. *Let G be either the Boolean graph B_n or its complement graph $\overline{B_n}$, and let Δ be the clique complex of the graph \overline{G} . Then*

- (1) *Each skeleton complex $\Delta^{(0,s)}$ of Δ is vertex decomposable.*
- (2) *Each pure skeleton complex $\Delta^{(s,s)}$ of Δ is pure shellable, thus Cohen-Macaulay.*

Note that each skeleton complex $\Delta^{(0,s)}$ of Δ is vertex decomposable if Δ is vertex decomposable, by [18, Lemma 3.10].

Recall that a *2-flag complex* is a complex Δ such that each minimal nonface of Δ has cardinality 2. Recall that a complex is a 2-flag complex if and only if Δ is a clique complex of a graph ([6, Proposition 9.1.3]). Note that the Alexander dual Δ^\vee of a 2-flag complex is pure of dimension $|V(\Delta)| - 2$.

Proposition 3.3. *Let G be either the Boolean graph B_n or its complement graph $\overline{B_n}$, and let Δ be the clique complex of the graph \overline{G} . Then the Alexander dual complex Δ^\vee is not shellable when $n \geq 4$.*

Proof: Recall that a complex Δ is shellable if there is a shelling order of the facets F_1, F_2, \dots, F_t such that for all i and k with $1 \leq i < k \leq t$, there exist $1 \leq j < k$ and $x \in F_k$, such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{x\}$. In the following, assume $n \geq 4$.

- (1) Let Δ be the clique complex of $\overline{B_n}$. Clearly,

$$\mathcal{F}(\Delta^\vee) = \{V \setminus \{a, b\} | a \in V(B_n), b \in V(B_n), a \cap b = \emptyset\},$$

where $V = 2^{[n]} \setminus \{[n], \emptyset\}$.

Since $n \geq 4$, we can choose $a, b \in V(B_n)$, say, $a = \{1, 2\}, b = \{2, 3\}$, such that

$$u \cap v \neq \emptyset, \forall u \neq v^c, u, v \in \{a, b, a^c, b^c\}.$$

Let $F_i = V \setminus \{a, a^c\}$, $F_k = V \setminus \{b, b^c\}$.

If assume that Δ^\vee is shellable, we can assume $F_i < F_k$ in the shelling of facets. Then by definition, there exist $1 \leq j < k$ and $x \in F_k$, such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{x\}$. If let $F_j = V \setminus \{c, d\}$, then we have the following two facts:

(i) $F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{x\}$, i.e.,

$$V \setminus \{a, a^c, b, b^c\} \subseteq V \setminus \{c, d, b, b^c\} = V \setminus \{x, b, b^c\}.$$

It follows that $\{x, b, b^c\} = \{c, d, b, b^c\}$ and $\{c, d, b, b^c\} \subseteq \{a, a^c, b, b^c\}$

(ii) $x \notin F_i$ and $x \notin F_j$, in which $F_i = V \setminus \{a, a^c\}$ and $F_j = V \setminus \{c, d\}$.

By (ii), $x \notin F_i = V \setminus \{a, a^c\}$, thus $x \in \{a, a^c\}$. Assume $x = a$, and assume further $c = a$ by fact (i). Then $d \in \{b, b^c\}$ since $\{x, b, b^c\} = \{c, d, b, b^c\}$. But then $c \cap d \neq \emptyset$ by the choice of a and b , contradicting to the assumption that F_j is a facet of Δ^\vee . The contradiction shows that Δ^\vee is not shellable, thus the edge ideal $I(B_n)$ does not have linear quotients.

(2) As for the clique complex Δ of B_n , clearly

$$\mathcal{F}(\Delta^\vee) = \{V \setminus \{a, b\} | a \in V(B_n), b \in V(B_n), a \cap b \neq \emptyset\}.$$

When $n \geq 4$, we can take $a, b, c, d \in V(B)$ with

$$a \cap b = \emptyset, a \cap c = \emptyset = a \cap d, b \cap c = \emptyset = b \cap d,$$

and consider

$$F_i = V \setminus \{a, b\}, F_k = V \setminus \{c, d\}.$$

If Δ^\vee is shellable, we can assume $F_i < F_k$ in the shelling of facets. Then a similar discussion leads to a contradiction. The details will be omitted. ■

We end this section by posing the following unsettled questions:

Question 3.4. *Let G be either the Boolean graph B_n or its complement graph $\overline{B_n}$, and let Δ be the clique complex of the graph \overline{G} .*

- (1) *Are the pure skeleton complexes $\Delta^{(s,s)}$ of Δ vertex decomposable?*
- (2) *Is Δ strongly shellable?*

4 Blow up of Boolean graphs and unmixed property

Recall that to get a finite *blow-up graph* G_T of a finite graph G is to replace every vertex v of G by a finite set T_v to get a possibly new and larger graph G_T , where $v \in T_v$. The induced subgraph of G_T on T_v is a discrete graph, while for distinct vertices u, v of G , u is adjacent to v in G if and only if each vertex of T_u is adjacent to all vertices of T_v in G_T , see [10, 13] for details.

If we further let T_v be a complete graph, then G_T becomes an *expanding* graph G_E of G ([11]). For a graph G , let \overline{G} be the complement graph of G in a complete graph with vertex set $V(G)$. Then the following observation holds true:

A graph H is a blow up of a graph G if and only if \overline{H} is an expanding graph of the graph \overline{G} .

Note that in a non-discrete Cohen-Macaulay bipartite graph, there exists an end vertex. Clearly, graph blow up does not keep anyone of the following properties of the original graph: chordal, vertex decomposable, Cohen-Macaulay. On the other hand, expanding a graph keeps a lot of properties unchanged, e.g, chordal, vertex decomposable, shellable, see [11] for some further discussion. Actually, for a graph, the result for chordal follows directly from Theorem 1.7, while that for vertex decomposable follows from Definition 1.4.

In general, a blow up of a Boolean graph is not unmixed. For example, the complete bipartite graph $K_{m,n}$ is a blow up of the Boolean graph B_2 and, it is unmixed if and only if $m = n$.

Example 4.1. *Let G_T be a finite blow up of the graph B_n . For any vertex $u \in B_n$, let $x_u = |T_u|$. Then*

- (1) *For $n = 2$, G_T is unmixed if and only if $G_T = K_{m,m}$ for some $m \geq 1$.*
- (2) *For $n = 3$, G_T is unmixed if and only if $x_i = x_{jk}$, $\forall \{i, j, k\} = [3]$.*
- (3) *For $n = 4$, G_T is unmixed if and only if the following seven equalities hold true:*

$$x_i = x_{jkl}, x_{ij} = x_{kl}, \forall \{i, j, k, l\} = [4].$$

- (4) *The Boolean graphs B_2 , B_3 and B_4 are unmixed.*

Proof: First, note the following observations: If a graph G contains a clique K of r vertices, then any minimal vertex cover of G contains at least $r - 1$ vertices of K ; also, G_T has a minimal vertex cover which contains $\cup_{i=1}^n T_i$.

- (i) For $n = 2$, the result is clear.
- (ii) For $n = 3$, consider the following four minimal vertex covers of G_T :

$$T_1 \cup T_2 \cup T_3, T_i \cup T_j \cup T_{\{i,j\}} (1 \leq i < j \leq 3).$$

Clearly, G_S is unmixed if and only if the vector $(x_1, x_2, x_3, x_{11}, x_{22}, x_{33})$ is the positive solution in \mathbb{Z}^6 of the following system of equations:

$$\begin{cases} x_1 + x_2 + x_{12} = x_1 + x_3 + x_{13} \\ x_1 + x_2 + x_{12} = x_2 + x_3 + x_{23} \\ x_1 + x_2 + x_{12} = x_1 + x_2 + x_3 \end{cases} \quad (1)$$

Then the result follows. In particular, it shows that the Boolean graph B_3 is unmixed.

(iii) For $n = 4$, note that $(\cup_{i=1}^4 T_i) \cup (\cup_{i=1}^3 T_{u_i})$ is a minimal vertex cover of G_T , where u_1, u_2, u_3 are taken from distinct $\{ij, kl\}$ with $\{i, j, k, l\} = [4]$ respectively. There are totally eight such minimal vertex covers of G_T . Also, there are four others, and one representative of them is

$$(\cup_{i=2}^4 T_i) \cup T_{234} \cup T_{23} \cup T_{24} \cup T_{34}.$$

Like the $n = 3$ case, it follows from the system of linear equations that $x_i = x_{jkl}$ holds for all $\{i, j, k, l\} = [4]$. Then it follows easily $x_{ij} = x_{kl}$.

The converse holds clearly.

In particular, the Boolean graph B_i ($1 \geq i \leq 4$) is unmixed. ■

This shows another way for illustrating Theorem 2.1. When n is large, things will become complicated. But a similar careful discussion shows that the unmixedness of the blow up G_T of the Boolean graph B_n ($n = 5, 6, 7$, respectively) amounts to the solving of a system of linear equations with indeterminate labeled properly according to their position in the layers.

The above example shows that *graph blow up* is a good concept for discussing unmixed property of graphs. We can even generalize it a little to obtain a *finite generalized blow up* G_S of a finite graph G explained in what follows. For every vertex v of G , let S_v be a disjoint union of S_{1v} with S_{2v} , in which $v \in S_{1v}$. Replace v by S_v to get a possibly new and larger graph G_S : For any $u \in V(G)$, the induced subgraph of G_S on each S_u is a discrete graph, while for distinct vertices u, v of G , u is adjacent to v in G iff each vertex of S_{1u} is adjacent to all vertices of S_v and each vertex of S_{1v} is adjacent to all vertices of S_u . Note that whenever none of S_{2u}, S_{2v} is empty, no vertices in S_{2u} is adjacent to a vertex in S_{2v} . By the definition, each blow up is a generalized blow up, of a graph; but the converse is clearly not true.

Generalized blow up occur naturally when we consider deleting a vertex from the graph B_n , as the following example shows.

Example 4.2. $B_n \setminus n \setminus 12 \dots n-1$ is a generalized blow up of B_{n-1} .

Proof: Clearly, the vertex $12 \dots n-1$ is isolated in the graph $B_n \setminus n$.

Let $G = B_n \setminus n \setminus 12 \dots n-1$. Then the vertex set of $V(G)$ splits with two parts, $\{A, A \cup \{n\}\}, \forall A \in V(B_{n-1})$. Thus if we add $A \cup \{n\}$ to the vertex A as the second part, then clearly, G is a generalized blow up of B_{n-1} , where for each vertex v of B_{n-1} , we have $|S_{1v}| = |S_{2v}| = 1$. ■

We end the paper with an easy discussion on the unmixedness of a generalized blow up of the graph $G = B_n$.

Example 4.3. Let G_S be a generalized blow up of the graph $G = B_2$. Then G_S is unmixed if and only if either $G_S = K_{m,m}$ or $|S_{11}| = |S_{12}| = |S_{21}| = |S_{22}|$.

Proof: Assume that G_S is a generalized blow up of the graph $G = B_2$, but not a blow up of B_2 . Note that $S_{1\{1\}} \cup S_{1\{2\}}$, $S_{2\{1\}} \cup S_{2\{2\}}$ and $S_{1\{1\}} \cup S_{2\{1\}}$ are minimal vertex covers of the graph G_S . Thus if G_S is unmixed, then we have

$$|S_{1\{1\}}| + |S_{1\{2\}}| = |S_{2\{1\}}| + |S_{2\{2\}}| = |S_{1\{1\}}| + |S_{2\{1\}}|$$

hence all $|S_{ij}|$ are identical.

The converse holds clearly.

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